A collection of mathematical formulas involving π

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Abstract

This note presents a collection of mathematical formulas involving the mathematical constant π.

1   Background

The mathematical constant known as $\pi = 3.14159\ldots$ is undeniably the most famous mathematical constant. Mathematicians since the days of Euclid and Archimedes have analyzed its properties and computed its numerical value.

This is a collection of many of formulas that have been established by mathematicians over the years involving π. Preference is given in this list for formulas that satisfy the following criteria:

- Formulas that give π (or a very simple expression involving π) explicitly, as opposed to implicit relations such as $e^{i\pi} + 1 = 0$.

- Formulas that give π (or a very simple expression involving π) as a straightforward infinite series or definite integral, as opposed to iterative algorithms or other multiple-formula schemes.

- Formulas that involve only simple notation (such as binomial coefficients, square roots, logs, summations, integrals, etc.) that would be familiar to anyone who has completed a beginning course in calculus.

- Formulas that are relatively new (discovered say within the last 15 years) or have current research interest.

Included in this listing are several classical infinite series for π that have actually have been used in historical calculations of π (both before and since the invention of the computer). This includes formulas 2, 3 and 4 prior to the 20th century, and formulas 5 and 6 in the latter part of the 20th century (in addition to some iterative algorithms not shown here).

Formulas 7 through 12 have the intriguing property that they permit digits (in certain specific bases) of the constant specified on the left-hand side to be calculated beginning at an arbitrary starting position, without having to calculate any of the digits that came before, by means of a relatively simple algorithm. Indeed, Formulas 7 and 8 have been used in computations of binary digits of π by several researchers. In the most recent such computation, binary digits of π beginning at position 2×10^{15} were calculated. Details are provided in [8, Sec 3.4–3.6]. Numerous similar recently-discovered formulas that posses the arbitrary digit-computation property for various mathematical constants are catalogued in [2].

Many of these formulas are quite new, in the sense that they were discovered only in recent years. The formulas mentioned in the previous paragraph are certainly in this category, having been discovered only in

the past 10–15 years. Many of the formulas from 13 through 46 were not well known until recently. Formulas 57 through 60 are also relatively new, in the sense that they are part of a class of integral formulas that are the subject of current research [3, 4, 5].

2   Credits

- Formula 1 was discovered by Leibniz and Gregory in the 1600s. Formula 2 was attributed to Euler in 1738. Formula 3 was discovered about the same time by Machin [8, pg. 105]. Formula 4 was discovered by Newton in the 1600s.

- Formula 5 is due to Ramanujan, and was used by Gosper in 1986 to compute π to over 17 million digits. The similar but more complicated Formula 6 is due to David and Gregory Chudnovsky, and was used by them to compute π to over one billion decimal digits [8, pg. 108].

- Formula 7 is known as the "BBP" formula for π, named for the initials of the co-authors of the 1997 paper where it was first presented [6]. It was discovered by numerical methods, in particular by applying the "PSLQ" algorithm of mathematician-sculptor Helaman Ferguson [13, 7]. Formula 8 is a variant of the BBP formula due to Bellard [8, pg. 124]. Formula 9 was found by Helaman Ferguson.

- Formula 10 appeared in [6]. Formulas 11 and 12 are due to David Broadhurst [11].

- Some of the summation formulas involving factorials and combinatorial coefficients (i.e., formulas 13 through 45) were found by Ramanujan; others are due to David and Gregory Chudnovsky. The Chudnovskys had these and many other formulas of this general type inscribed on the floor of their research center at Brooklynn Polytechnic University in New York City [12]. Four exceptions are Formula 31, which is due to Ramanujan but appeared in [10, pg. 188], Formulas 40 and 41, which are due to Guillera [14], and 46, which is due to Almkvist and Guillera [1].

- Formulas 47 through 58 have been known for some time; they are from [9, pg. 5, 48, 320–321].

- Formula 59 is an example of numerous formulas of this general type recently discovered by computational methods, typically involving the PSLQ algorithm [13, 7], in studies of Ising theory in mathematical physics [3]. Formulas 60, 61 and 62 are examples of recent discoveries, also by computational methods involving the PSLQ algorithm, in the theory of box integrals [4, 5]. Formula 59, for instance, can be thought of as specifying the average distance between two points in the unit 3-cube.

3   Formulas

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} \tag{1}$$

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)2^{2n+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^{2n+1}} \tag{2}$$

$$\frac{\pi}{4} = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)5^{2n+1}} - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)239^{2n+1}} \tag{3}$$

$$\pi = \frac{3\sqrt{3}}{4} - 24\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+3)(2n-1)4^{2n+1}} \tag{4}$$

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!(1103 + 26390n)}{(n!)^4 396^{4n}} \tag{5}$$

$$\frac{1}{\pi} = 12 \sum_{n=0}^{\infty} \frac{(-1)^n (6n)!(13591409 + 545140134n)}{(3n)!(n!)^3 640320^{3n+3/2}} \tag{6}$$

$$\pi = \sum_{n=0}^{\infty} \frac{1}{16^n} \left(\frac{4}{8n+1} - \frac{2}{8n+4} - \frac{1}{8n+5} - \frac{1}{8n+6} \right) \tag{7}$$

$$\pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n (2n+1)} - \frac{1}{64} \sum_{n=0}^{\infty} \frac{(-1)^n}{1024^n} \left(\frac{32}{4n+1} + \frac{8}{4n+2} + \frac{1}{4n+3} \right) \tag{8}$$

$$\pi = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n} \left(\frac{2}{4n+1} + \frac{2}{4n+2} + \frac{1}{4n+3} \right) \tag{9}$$

$$\pi^2 = \frac{9}{8} \sum_{n=0}^{\infty} \frac{1}{64^n} \left(\frac{16}{(6n+1)^2} - \frac{24}{(6n+2)^2} - \frac{8}{(6n+3)^2} - \frac{6}{(6n+4)^2} + \frac{1}{(6n+5)^2} \right) \tag{10}$$

$$\pi\sqrt{3} = \frac{1}{9} \sum_{n=0}^{\infty} \frac{1}{729^n} \left(\frac{81}{12n+1} - \frac{54}{12n+2} - \frac{9}{12n+4} - \frac{12}{12n+6} - \frac{3}{12n+7} - \frac{2}{12n+8} - \frac{1}{12n+10} \right) \tag{11}$$

$$\pi^2 = \frac{2}{27} \sum_{n=0}^{\infty} \frac{1}{729^n} \left(\frac{243}{(12n+1)^2} - \frac{405}{(12n+2)^2} - \frac{81}{(12n+4)^2} - \frac{27}{(12n+5)^2} - \frac{72}{(12n+6)^2} \right.$$
$$\left. - \frac{9}{(12n+7)^2} - \frac{9}{(12n+8)^2} - \frac{5}{(12n+10)^2} + \frac{1}{(12n+11)^2} \right) \tag{12}$$

$$3\pi + 8 = \sum_{n=0}^{\infty} \frac{12n 2^{2n}}{\binom{4n}{2n}} \tag{13}$$

$$\frac{\pi^2}{6} - 2\log^2 2 = \sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{n^2 4^n} \tag{14}$$

$$15\pi + 52 = \sum_{n=0}^{\infty} \frac{(126n^2 - 24n + 8)2^{3n}}{\binom{6n}{3n}} \tag{15}$$

$$105\pi + 304 = \sum_{n=0}^{\infty} \frac{(1920n^3 - 928n^2 + 424n - 16)2^{4n}}{\binom{8n}{4n}} \tag{16}$$

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}^3 (42n+5)}{16^{3n+1}} \tag{17}$$

$$16\pi\sqrt{3} + 81 = \sum_{n=0}^{\infty} \frac{(49n+1)8^n}{3^n \binom{3n}{n}} \tag{18}$$

$$162 - 6\pi\sqrt{3} - 18\log 3 = \sum_{n=0}^{\infty} \frac{(-245n + 338)8^n}{3^n \binom{3n}{n}} \tag{19}$$

$$\pi = \sum_{n=0}^{\infty} \frac{(50n-6)}{2^n \binom{3n}{n}} \tag{20}$$

$$15\pi + 42 = \sum_{n=1}^{\infty} \frac{(-4)^n (2n)!^2 (3n)!(201 - 952n)}{(6n)!n!} \tag{21}$$

$$15\pi\sqrt{2} + 27 = \sum_{n=0}^{\infty} \frac{8^n (2n)!^2 (3n)!(350n - 17)}{(6n)!n!} \tag{22}$$

$$40\pi\sqrt{3} + 243 = \sum_{n=1}^{\infty} \frac{(-27)^n (2n)!^2 (3n)!(81 - 1080n)}{(6n)!n!} \tag{23}$$

$$20\pi\sqrt{3} + 89 = \sum_{n=1}^{\infty} \frac{(-1/3)^n (2n)!^2 (3n)!(4123 - 22100n)}{(6n)!n!} \tag{24}$$

$$15\pi + 240\log 2 - 528 = \sum_{n=1}^{\infty} \frac{(-1/2)^n (89012n^3 - 77362n^2 + 482n + 3028)}{\binom{5n}{2n}} \tag{25}$$

$$24516 - 360\pi\sqrt{3} = \sum_{n=1}^{\infty} \frac{9^n (2743n^2 - 130971n - 12724)}{\binom{4n}{n}} \tag{26}$$

$$45\pi + 1164 = \sum_{n=1}^{\infty} \frac{8^n (430n^2 - 6240n - 520)}{\binom{4n}{n}} \tag{27}$$

$$40\pi\sqrt{3} + 1872 = \sum_{n=0}^{\infty} \frac{3^n (7175n^2 - 15215n + 480)}{\binom{4n}{n}} \tag{28}$$

$$288\pi\sqrt{3} - 576\log 2 + 324 = \sum_{n=0}^{\infty} \frac{(9/8)^n (5692 + 6335n - 5415n^2)}{\binom{4n}{n}} \tag{29}$$

$$1008\pi\sqrt{3} - 576\log 2 + 7587 = \sum_{n=0}^{\infty} \frac{(9/8)^n (7517 + 1145n + 18050n^2)}{\binom{4n}{n}} \tag{30}$$

$$\frac{16}{\pi} = \sum_{n=0}^{\infty} \frac{42n + 5}{4096^n} \binom{2n}{n}^3 \tag{31}$$

$$\frac{4}{\pi} = \sum_{n=0}^{\infty} \frac{(-1)^n (4n)!(20n + 3)}{4^{4n}(n!)^4 2^{2n+1}} \tag{32}$$

$$\frac{4}{\pi} = \sum_{n=0}^{\infty} \frac{(-1)^n (4n)!(260n + 23)}{4^{4n}(n!)^4 18^{2n+1}} \tag{33}$$

$$\frac{4}{\pi} = \sum_{n=0}^{\infty} \frac{(-1)^n (4n)!(21460n + 1123)}{4^{4n}(n!)^4 882^{2n+1}} \tag{34}$$

$$\frac{2}{\pi\sqrt{3}} = \sum_{n=0}^{\infty} \frac{(4n)!(8n + 1)}{4^{4n}(n!)^4 3^{2n+1}} \tag{35}$$

$$\frac{1}{2\pi\sqrt{2}} = \sum_{n=0}^{\infty} \frac{(4n)!(10n + 1)}{4^{4n}(n!)^4 9^{2n+1}} \tag{36}$$

$$\frac{4}{\pi\sqrt{3}} = \sum_{n=0}^{\infty} \frac{(-1)^n (4n)!(28n + 3)}{4^{4n}(n!)^4 3^n 4^{2n+1}} \tag{37}$$

$$\frac{4}{\pi\sqrt{5}} = \sum_{n=0}^{\infty} \frac{(-1)^n (4n)!(644n + 41)}{4^{4n}(n!)^4 5^n 72^{2n+1}} \tag{38}$$

$$\frac{1}{3\pi\sqrt{3}} = \sum_{n=0}^{\infty} \frac{(4n)!(40n + 3)}{4^{4n}(n!)^4 49^{2n+1}} \tag{39}$$

$$\frac{32}{\pi^2} = \sum_{n=0}^{\infty} \frac{\binom{4n}{2n}\binom{2n}{n}^4 (120n^2 + 34n + 3)}{2^{16n}} \tag{40}$$

$$\frac{128}{\pi^2} = \sum_{n=0}^{\infty} \frac{(-1)^n \binom{2n}{n}^5 (820n^2 + 180n + 13)}{2^{20n}} \tag{41}$$

$$\frac{2}{\pi} = \sum_{n=0}^{\infty} (-1)^n \binom{2n}{n}^3 \frac{4n + 1}{64^n} \tag{42}$$

$$\frac{4}{\pi} = \sum_{n=0}^{\infty} \binom{2n}{n}^3 \frac{6n + 1}{256^n} \tag{43}$$

$$\frac{16}{\pi} = \sum_{n=0}^{\infty} \binom{2n}{n}^3 \frac{42n + 5}{4096^n} \tag{44}$$

$$\pi + 4 = \sum_{n=0}^{\infty} \frac{2^{n+1}}{\binom{2n}{n}} \tag{45}$$

$$\frac{6}{\pi^2} = 64 \sum_{n=0}^{\infty} \frac{(6n)!(532n^2 + 126n + 9)}{(n!)^6 10^{6n+3}} \tag{46}$$

$$\frac{22}{7} - \pi = \int_0^1 \frac{x^4(1 - x)^4 \mathrm{d}x}{1 + x^2} \tag{47}$$

$$\frac{\pi}{8} = \int_0^1 \frac{x^2 \mathrm{d}x}{(1 + x^4)\sqrt{1 - x^4}} \tag{48}$$

$$\frac{\pi(1 + 2\log 2)}{8} = \int_0^{\infty} x e^{-x} \sqrt{1 - e^{-2x}} \, \mathrm{d}x \tag{49}$$

$$4\pi \log^2 2 + \frac{\pi^3}{3} = \int_0^{\infty} \frac{x^2 \, \mathrm{d}x}{\sqrt{e^x - 1}} \tag{50}$$

$$\pi \log 2 = \int_0^{\pi/2} \frac{x^2 \, \mathrm{d}x}{\sin^2 x} \tag{51}$$

$$\frac{\pi^3}{24} + \frac{\pi \log^2 2}{2} = \int_0^{\pi/2} \log^2(\cos x) \, \mathrm{d}x \tag{52}$$

$$\frac{8\pi^3}{81\sqrt{3}} = \int_0^1 \frac{\log^2 x \, \mathrm{d}x}{x^2 + x + 1} \tag{53}$$

$$\frac{\pi}{2} - \log 2 = \int_0^1 \frac{\log(1 + x^2) \, \mathrm{d}x}{x^2} \tag{54}$$

$$\frac{2\pi \log 3}{\sqrt{3}} = \int_0^1 \frac{\log(1 + x^3) \, \mathrm{d}x}{1 - x + x^2} \tag{55}$$

$$\frac{\pi^2}{6} = \int_0^1 \int_0^1 \frac{\mathrm{d}x\,\mathrm{d}y}{1 - xy} \tag{56}$$

$$\frac{\pi^2}{8} = \int_0^1 \int_0^1 \frac{\mathrm{d}x\,\mathrm{d}y}{1 - x^2 y^2} \tag{57}$$

$$\frac{-\pi}{6} + \log(2 + \sqrt{3}) = \int_0^1 \int_0^1 \frac{\mathrm{d}x\,\mathrm{d}y}{\sqrt{1 + x^2 + y^2}} \tag{58}$$

$$5 - \pi^2 - 4\log 2 + 16\log^2 2 = \int_0^1 \int_0^1 \left(\frac{x-1}{x+1}\right)^2 \left(\frac{y-1}{y+1}\right)^2 \left(\frac{xy-1}{xy+1}\right)^2 \mathrm{d}x\,\mathrm{d}y \tag{59}$$

$$-\frac{\pi}{4} + \frac{3\log(2 + \sqrt{3})}{2} = \int_0^1 \int_0^1 \int_0^1 \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{\sqrt{x^2 + y^2 + z^2}} \tag{60}$$

$$-\frac{\pi}{24} + \frac{\sqrt{3}}{4} + \frac{\log(2 + \sqrt{3})}{2} = \int_0^1 \int_0^1 \int_0^1 \sqrt{x^2 + y^2 + z^2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{61}$$

$$-\frac{\pi}{60} + \frac{2\sqrt{3}}{5} + \frac{7\log(2 + \sqrt{3})}{20} = \int_0^1 \int_0^1 \int_0^1 (x^2 + y^2 + z^2)^{3/2}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{62}$$

References

[1] Gert Almkvist and Jesus Guillera, "Ramanujan-like series for $1/\pi^2$ and String Theory," 27 Sep 2010, available at `http://arxiv.org/abs/1009.5202`.

[2] David H. Bailey, "A Compendium of BBP-Type Formulas for Mathematical Constants," updated 14 Oct 2010, available at `http://crd.lbl.gov/~dhbailey/dhbpapers/bbp-formulas.pdf`.

[3] David H. Bailey, Jonathan M. Borwein and Richard E. Crandall, "Integrals of the Ising Class," *Journal of Physics A: Mathematical and General*, vol. 39 (2006), pg. 12271–12302.

[4] David H. Bailey, Jonathan M. Borwein and Richard E. Crandall, "Box Integrals," *Journal of Computational and Applied Mathematics*, vol. 206 (2007), pg. 196–208.

[5] David H. Bailey, Jonathan M. Borwein and Richard E. Crandall, "Advances in the Theory of Box Integrals," *Mathematics of Computation*, vol. 79, no. 271 (Jul 2010), pg. 1839–1866.

[6] David H. Bailey, Peter B. Borwein and Simon Plouffe, "On the Rapid Computation of Various Polylogarithmic Constants," *Mathematics of Computation*, vol. 66, no. 218 (Apr 1997), pg. 903–913.

[7] David H. Bailey and David J. Broadhurst, "Parallel Integer Relation Detection: Techniques and Applications," *Mathematics of Computation*, vol. 70, no. 236 (Oct 2000), pg. 1719–1736.

[8] Jonathan M. Borwein and David H. Bailey, *Mathematics by Experiment: Plausible Reasoning in the 21st Century*, AK Peters, Natick, MA, 2008.

[9] Jonathan M. Borwein, David H. Bailey and Roland Girgensohn, *Experimentation in Mathematics: Computational Paths to Discovery*, AK Peters, Natick, MA 2004.

[10] Jonathan M. Borwein and Peter B. Borwein, *Pi and the AGM: A Study in Analytic Number Theory and Computational Complexity*, CMS Series of monographs and Advanced Books in Mathematics, John Wiley, Hoboken, NJ, 1987.

[11] David J. Broadhurst, "Massive 3-loop Feynman diagrams reducible to SC* primitives of algebras of the sixth root of unity," manuscript, 1998, available at `http://www.arxiv.org/abs/hep-th/9803091`.

[12] David and Gregory Chudnovsky, "Listing of Ramanujan-type formulas," copy in author's possession, 2000.

[13] Helaman R. P. Ferguson, David H. Bailey and Stephen Arno, "Analysis of PSLQ, An Integer Relation Finding Algorithm," *Mathematics of Computation*, vol. 68, no. 225 (Jan 1999), pg. 351–369.

[14] Jesus Guillera, "Some binomial series obtained by the WZ-method," *Advances in Applied Mathematics*, vol. 29 (2002), pg. 599–603.